Indian Statistical Institute, Bangalore

B. Math. III Second Semester

Differential Geometry II: Final Exam

Duration: 3 hours **Date : May 08, 2015**

Maximum Marks: 50

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(1) *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth function. Let $q \in \mathbb{R}$ be a regular value for f such that $S = f^{-1}\{q\} \neq \emptyset$.*

i) *Prove that S is a smooth manifold of dimension $n-1$*

ii) *Let $p \in S$. Show that the tangent space $T_pS = Ker Df(p)$.*

iii) *Calculate T_pS^n, where $S^n \subset \mathbb{R}^{n+1}$ is the standard unit sphere.*

iv) *Show that the tangent bundle of the sphere $S^n \subset \mathbb{R}^{n+1}$ is the set*

$$\{(x, v) \in S^n \times \mathbb{R}^{n+1} :< v, x >= 0\}.$$

where $< ., . >$ is the standard inner product in $\mathbb{R}^{n+1}$.

(20 marks)

(2) i) *Compute the Lie bracket $[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}]$ on $\mathbb{R}^2$.*

ii) *Let S be a submanifold of $\mathbb{R}^n$ of dimension $k < n$. Is it true that the Lie bracket of any two vector fields of S is a vector field of S?*

(10 marks)

(3) *Let ω be the two form on $S^2 \subset \mathbb{R}^3$ given by*

$$\omega = \begin{cases} \frac{dy \wedge dz}{x}, & x \neq 0 \\ \frac{dz \wedge dx}{y}, & y \neq 0 \\ \frac{dx \wedge dy}{z}, & z \neq 0. \end{cases}$$

Calculate $\int_{S^2} \omega$.

(10 marks)

(4) i) *Calculate the Riemannian metric of S^2 induced from $\mathbb{R}^3$.*

ii) *Calculate the Levi-Civita connection of S^2 with above mentioned Riemannian metric.*

ii) *Calculate the curvature tensor in this case.*

(20 marks)

Note: You can use well-known theorems taught in the class, but you need to write precise statement of the theorem you are using.